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A THEORETICAL ANALYSIS ON STOCHASTIC BEHAVIOUR OF SHORT CRACKS AND FATIGUE DAMAGE OF METALS

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ABSTRACT In this paper, the closed form of solution to the stochastic differential equation for a fatigue crack evolution system is derived, and the relationship between metal fatigue damage and crack stochastic behaviour is investigated. It is found that the damage extent of metals is independent of crack stochastic behaviour if the stochastic deviation of the crack growth rate is directly proportional to its mean value. The evolution of stochastic deviation of metal fatigue damage in the stage close to the transition point between short and long crack regimes is also discussed.

KEY WORDS short crack, crack numerical density, stochastic behaviour, fatigue damage, crack growth rate

I. INTRODUCTION

Experimental investigations have revealed that the characteristics of fatigue damage of various metallic alloys in the primary and the final stage of fatigue failure possess different mechanisms of fatigue cracking. In the primary stage of fatigue failure, the length of most cracks is comparable with the grain size of alloy concerned, and the fatigue damage cumulation is controlled by a large number of dispersed short cracks, which presents collective damage characteristics with the gradual evolution of crack numerical density[1]. The method based on the equilibrium of crack numerical density[2] is a possible approach to describing and analyzing the collective evolution process of dispersed short cracks[2,3]. The basic conception of the method is that the total number of dispersed cracks is determined by the effects of crack nucleation and crack growth. The equilibrium equation is:

$$\frac{\partial}{\partial t}n(c,t) + \frac{\partial}{\partial c}[A(c)n(c,t)] = N_p n_N(c) \tag{1}$$

where $n(c,t)$ is the crack numerical density, with $n(c,t)dl$ being the number of cracks with

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length between c and $c+dc$ at time t. $A(c)$ is crack growth rate, and $n_N(c)$ is crack nucleation rate. All of these physical quantities are dimensionless, and the nondimensional coefficient $N_s = n_N^* d/(n^* A^*)$, with n_N^* being the characteristic crack nucleation rate, A^* the characteristic crack growth rate, n^* the characteristic crack numerical density, and d the characteristic dimension of the material concerned (e. g. the grain diameter), noting that d/A^* is the characteristic time. Equation (1) describes the equilibrium of crack numerical density in phase space. The second term at the left side describes the flow of crack numerical density in the phase space, namely, the contribution to crack numerical density made by crack growth, and the term at the right side describes the effect of crack nucleation.

Letting $n(c,0)=0$ at the initial stage, and assuming that the threshold value of crack length for growth is zero, we have the theoretical solution of Eq. (1)[2]:

$$n(c,t) = \frac{1}{A(c)} \int_{\eta(c,t)}^{c} N_s n_N(c')\mathrm{d}c' \tag{2}$$

The lower integral boundary $\eta(c,t)$ is of such a definition that for a crack with an initial length of $\eta(c,t)$ at $t=0$, its length will grow to c at time t at the growth rate of $A(c)$. Figure 1 shows the evolution process of $n(c, t)$ calculated from Eq. (1), which is obtained after the forms of $A(c)$ and $n_N(c)$ are given[3]. From the solution of Eq. (1) we find that $n(c,t)$ gradually tends to a stable curve $n_0(c)$:

$$n_0(c) = \frac{1}{A(c)} \int_{0}^{c} N_s n_N(c')\mathrm{d}c' \tag{3}$$

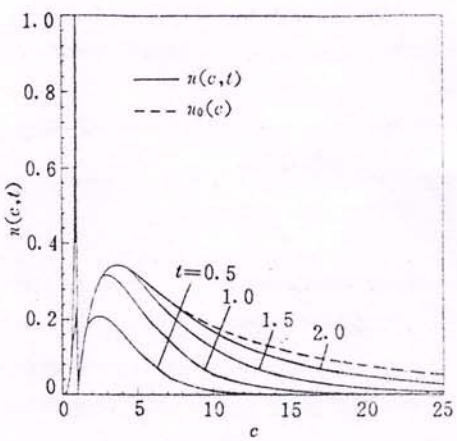

Fig. 1 Distribution of crack numerical density $n(c,t)$ at different values of t.

This is shown as the dash-line in Fig. 1.

If $A(c)$ and $n_N(c)$ in the above equations represent the mean value of a time series, the solutions of these equations are the results in average sense. In fact, there exists stochastic fluctuation of damage behaviour, and the extent of the fluctuation is a vital factor of damage evolution. In this paper, the stochastic fluctuation of crack numerical density, especially the correlation between the extent of stochastic fluctuation for the distribution of crack numerical density in the region with the crack length close to c_{cr}, is investigated. Here, c_{cr} denotes the critical crack length of the boundary between short and long crack regimes. The stochastic fluctuations of crack growth rate and crack nucleation rate are considered as two major aspects in relation to the behaviour of fatigue damage. The factors that dominate the stochastic fluctuation of fatigue damage are discussed.

Ⅱ. STOCHASTIC DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

There have been several studies concerning the stochastic deviation of crack growth rate[4~6], in which the relationship between the probability of single crack length and the stochastic deviation of crack growth rate has been investigated. However, in the short-crack damage regime which normally takes up a large portion of total fatigue life, the development of short cracks may present collective evolution process and the damage cumulation controlled by a large number of dispersed short cracks[3]. Therefore the relationship between stochastic fluctuation of crack numerical density and fatigue behaviour of dispersed short cracks should be studied.

In general, the stochastic behaviour of short cracks can be described by the stochastic deviation of crack growth rate $A(c)$ and crack nucleation rate $n_N(c)$. Hence, it is assumed that

$$\begin{aligned} A(c) &= A_0(c) + L(c)W(t) \\ n_N(c) &= n_{N0}(c) + H(c)W(t) \end{aligned} \tag{4}$$

where $A_0(c)$ is the expectation value (i. e. mean value) of crack growth rate; $n_{N0}(c)$ is the expectation value of crack nucleation rate; $W(t)$ is Gauss white noise; $L(c)$ and $H(c)$ are functions representing the relationship between crack length and crack stochastic behaviour.

Substituting Eq. (4) into Eq. (1), we have:

$$\frac{\partial n(c,t)}{\partial t} + \frac{\partial[A_0(c)n(c,t)]}{\partial c} = n_{N0}(c) + \left\{H(c) - \frac{\partial[L(c)n(c,t)]}{\partial c}\right\}W(t) \tag{5}$$

Equation (5) is the stochastic differential equation of fatigue damage evolution. By introducing a series of scattering points for continuous crack length c, Eq. (5) can be rewritten as:

$$\mathrm{d}n_j(t) = \left[-\frac{A_j}{\Delta c}n_j(t) + \frac{A_{j-1}}{\Delta c}n_{j-1}(t) + n_{N0j}\right]\mathrm{d}t + \left[H_j - \frac{L_j}{\Delta c}n_j(t) + \frac{L_{j-1}}{\Delta c}n_{j-1}(t)\right]\mathrm{d}B \tag{6}$$

where $j=1,2,3,\cdots$; $(\cdot)_j$ has the meaning of $\cdot(c_j)$, and $c_j=\Delta c\cdot j$, with Δc being a small step of crack length and B the Wiener process. According to Eq. (6) we can find that $n_j(t)$ can be derived if $n_{j-1}(t)$ is given. Because $n(t)|_{c=0}$ is known as the boundary condition, $n_j(t)(j=1,2,3,\cdots)$ can be obtained recursively. Therefore in the following, $n_{j-1}(t)$ will be used as a known function.

Equation (6) has the same form as Ito's formula[7]. If the range of stochastic fluctuation of $n(c,t)$ is relatively small compared with its mean value, the stochastic parts of $n_j(t)$ and $n_{j-1}(t)$ may be omitted. Thus Eq. (6) has the form:

$$\mathrm{d}n_j(t) = \left[-\frac{A_j}{\Delta c}\bar{n}_j(t) + \frac{A_{j-1}}{\Delta c}\bar{n}_{j-1}(t) + n_{N0j}\right]\mathrm{d}t + \left[H_j - \frac{L_j}{\Delta c}\bar{n}_j(t) + \frac{L_{j-1}}{\Delta c}\bar{n}_{j-1}(t)\right]\mathrm{d}B \tag{7}$$

where $\hat{n}_j(t)$ and $\hat{n}_{j-1}(t)$ are expectation values of $n_j(t)$ and $n_{j-1}(t)$ respectively, which can be solved according to Eq. (1).

By setting the initial condition of $n_j(0)=0$, Eq. (7) has the solution:

$$n_j(t) = \left[\exp\left(-\frac{A_j}{\Delta}t\right)\right]\left\{\int_0^t\left[\exp\left(\frac{A_j}{\Delta}s\right)\right]\left[\frac{A_{j-1}}{\Delta}\hat{n}_{j-1}(s) + n_{N0j}\right]\mathrm{d}s + \int_0^t\left[\exp\left(\frac{A_j}{\Delta}s\right)\right]\left[H_j - \frac{L_j}{\Delta}\hat{n}_j(s) + \frac{L_{j-1}}{\Delta}\hat{n}_{j-1}(s)\right]\mathrm{d}B_s\right\} \tag{8}$$

where the integration has the meaning of mean-square integration.

Writing $h_j(t)$ as the stochastic part of $n_j(t)$, from Eq. (8), we have:

$$h_j(t) = \left[\exp\left(-\frac{A_j}{\Delta}t\right)\right]\cdot\int_0^t\left[\exp\left(\frac{A_j}{\Delta}s\right)\right]\left[H_j - \frac{L_j}{\Delta}\hat{n}_j(s) + \frac{L_{j-1}}{\Delta}\hat{n}_{j-1}(s)\right]\mathrm{d}B_s \tag{9}$$

Equations (8) and (9) represent the dependence of crack numerical density upon the stochastic fluctuation of crack growth rate and crack nucleation rate.

Ⅲ. DISCUSSION

From Eq. (9), the average value $E\{h_j(t)\}$ and the variance $D\{h_j(t)\}$ due to the stochastic part of crack numerical density $h_j(t)$ can be shown:

$$E\{h_j(t)\} = \left[\exp\left(-\frac{A_j}{\Delta}t\right)\right]\int_0^t\left[\exp\left(\frac{A_j}{\Delta}s\right)\right]E\left\{\left[H_j - \frac{\partial(L\hat{n})}{\partial c}\bigg|_{c=c_j}\right]W_s\right\}\mathrm{d}s = 0 \tag{10}$$

and

$$D\{h_j(t)\} = \int_0^t\left\{\left[\exp\left(-\frac{A_j}{\Delta}(t-s)\right)\right]\left[H_j - \frac{\partial(L\hat{n})}{\partial c}\bigg|_{c=c_j}\right]\right\}^2\mathrm{d}s \tag{11}$$

It is common sense that the mean value of a stochastic deviation is always zero. As a consequence, $E\{h_j(t)\}$ is also always zero. Therefore the stochastic behaviour of crack numerical density is predominated by $D\{h_j(t)\}$. It can be seen from Eq. (11) that $D\{h_j(t)\}$ is a monotonically incremental function.

3.1 An Approximation for Stochastic Analysis

Taking into account the two-phase model[8] of fatigue process, we assume that

$$A_j(c) = c^{\nu} \tag{12}$$

and

$$L(c) = \alpha c^{\beta} \tag{13}$$

where ν, α and β are material parameters. The order of magnitude of α may be inferred from

published data of crack growth rate[9~11]: the values of α of different materials (pure nickel, Monel K500 Ni-Cu alloy and Inconel 625 Cr-Ni-Fe alloy) are within the range from 0.3 to 7.0.

Combining Eqs. (12), (13) and (2), we have

$$\frac{\partial(L\dot{n})}{\partial c} = \alpha(\beta-\nu)c^{\beta-\nu-1}\int_{\eta(c,s)}^{c} n_{N0}(c')\mathrm{d}c' + \alpha c^{\beta-\nu}n_{N0}(c)\left[1-\frac{\partial\eta(c,s)}{\partial c}\right] \tag{14}$$

where $\dot{n}$ has the same form as Eq. (2).

Noticing that a large part of fatigue life of metallic materials is in the regime of short crack damage, and that the critical time characterizing the transition of short and long crack regimes is mainly related to the cracks with the length close to c_{cr}[12], it is important to discuss the crack numerical density near the critical crack length c_{cr}. Because the crack nucleation rate tends to zero if the crack length is relatively large, we may assume that under this condition, h_j, the stochastic deviation of crack nucleation rate, also tends to zero, which suggests that the crack numerical density of the cracks with the length close to c_{cr} is independent of the stochastic behaviour of crack nucleation. Thus, substituting Eq. (14) into (11), we are able to derive the formula for $D\{h_j(t)\}$:

$$D\{h_j(t)\}|_{c>c_1} = \int_0^t\left\{\exp\left[-\frac{A_j}{\Delta c}(t-s)\right]\alpha(\beta-\nu)c_j^{\beta-\nu-1}\int_{\eta(c,s)}^{c_j} n_{N0}(c')\mathrm{d}c'\right\}^2\mathrm{d}s \tag{15}$$

where c_1 denotes the crack length in the region close to c_{cr}.

In correspondence with the stable curve shown in Fig. 1 (dash line), we take $\eta(c,t)\to 0$, and for simplicity, we write $\int_0^c n_{N0}(c')\mathrm{d}c' \equiv \kappa$, which denotes the total number of cracks nucleated in unit time. If h_j^* represents the corresponding stochastic deviation of crack numerical density, Eq. (15) is reduced to

$$\begin{aligned} D\{h_j^*\}|_{c>c_1} &= \frac{\Delta c}{2A_j}[\alpha\kappa(\beta-\nu)]^2 c_j^{2(\beta-\nu-1)}\left[1-\exp\left(-\frac{2A_j}{\Delta c}t\right)\right] \\ &\to \frac{\Delta c}{2A_j}[\alpha\kappa(\beta-\nu)]^2 c_j^{2(\beta-\frac{3}{2}\nu-1)} \end{aligned} \tag{16}$$

In the above derivation, we use the condition that $\exp\left(-\frac{2A_j}{\Delta c}t\right)$ tends to zero when the distribution of crack numerical density becomes stable. This is under the consideration that for the crack with the length close to c_{cr}, its growth rate is relatively large.

3.2 Stochastic Behaviour of Fatigue Damage at the Region near c_{cr}

Equations (15) and (16) represent the evolution of stochastic deviation of crack numerical

density for crack length near c_{cr}. They imply that the stochastic deviation of crack numerical density will increase with a decrease of the mean value of crack growth rate (A_j) or an increase of the stochastic deviation of crack growth rate (β).

When the evolution stage of fatigue damage is close to the transition point between short and long crack regimes, the fatigue behaviour is predominantly determined by relatively large cracks whose length is close to c_{cr}, and the extent of stochastic deviation of fatigue damage is described by $D\{h_j(t)\}$ of the largest crack. Note that the maximum crack length in a damage system increases with the progress of fatigue process. Thus the evolution of stochastic deviation of fatigue damage is characterized by Eq. (16). Figure 2 shows several typical cases of the solution of Eq. (16) at different values of β and ν. It is seen that $D\{h_j(t)\}$ decreases with an increase of the mean value of the crack growth rate; on the other hand, $D\{h_j(t)\}$ increases with an increase of the extent of stochastic deviation of the crack growth rate. When $\beta=3\nu/2+1$, the two effects offset each other. It is clear that, under the condition of $\beta>3\nu/2+1$, the extent of stochastic fluctuation of fatigue behaviour will increase with the evolution of fatigue damage.

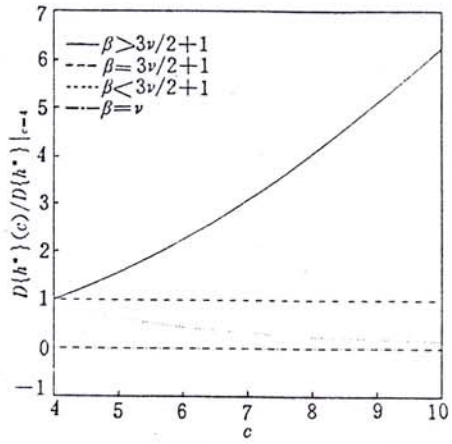

Fig. 2 Stochastic extent of crack numerical density versus crack length at different values of β and ν.

Considering that the range of stochastic deviation of crack growth rate is directly proportional to its mean value, i. e.

$$L(c) \propto A_j(c) \tag{17}$$

we have $\beta=\nu$. Based on Eqs. (15) and (16), it can be seen that under this condition, the variance of crack numerical density in the region of cracks with the length close to c_{cr} is zero, which is also shown in Fig. 2 (dash-dot line). Thus the difference in the fatigue behaviour between different specimens should be explained not by the crack stochastic behaviour, but by intrinsic material parameters such as grain size, inclusion dimension, initial defect, etc.

Ⅳ. CONCLUSIONS

In this paper, the stochastic differential equation of metal fatigue evolution is proposed and analyzed. The influence of the stochastic behaviour of crack growth and crack nucleation on the distribution of crack numerical density is discussed. The following conclusions are drawn:

(1) The stochastic deviation of crack numerical density becomes small with an increase in the mean value of the crack growth rate and with a decrease in the stochastic extent of crack

propensity.

(2) In the case of $\beta > 3\nu/2+1$, the extent of stochastic fluctuation for the fatigue crack behaviour of metallic materials increases with the progression of fatigue damage.

(3) If the range of stochastic deviation of the crack growth rate is in direct proportion to its mean value, the crack numerical density in the region near c_{cr} is independent of the stochastic deviation of the crack growth rate.

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