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A theoretical analysis on the dynamic cleavage cracking in a constant-*K* specimen

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Summary. Through an energy analysis of the cleavage cracking in a constant-K sample, the analytical solution of the dynamic fracture resistance for a brittle, homogeneous material is obtained. The dependence of the crack behavior on the accumulated continuum dissipation associated with the dynamic effect is analyzed. The relationship between the critical energy release rate of crack arrest and the applied stress intensity factor is discussed in detail.

1 Introduction

The dynamic crack propagation and crack arrest in brittle materials are of immense scientific interest and technological importance. As the crack propagates at a relatively high velocity, $\dot{a}$, with a being the crack length, stress waves are emitted from the tip into the background, accompanied by a considerable dissipation of kinetic energy associated with the dynamic effect that depends upon the nonlocal atomic interactions. As a result, the crack growth driving force changes, and the fracture resistance becomes a function of the crack size and the crack growth rate [1]. While the dominant energy dissipation mechanisms and processes are related to the crack-tip atomic structure, the overall energy balance can be analyzed in context of linear elastic fracture mechanics (LEFM), i.e., the Griffith theory [2]. When the energy release rate, G, equals the fracture resistance, R, a static crack starts to propagate. If $\partial G/\partial a > \partial R/\partial a$, the crack growth becomes unstable and is affected by the loading rate and the sample geometry [3].

Over years, the dynamic crack front advance has been observed experimentally and simulated numerically [4]–[6]. The governing equation can be stated as [7]

$$K(t) = K_{\mathrm{ID}}(\dot{a}), \tag{1}$$

where K is the dynamic stress intensity factor, K_{ID} is the dynamic fracture toughness, and t is the time. However, even in highly brittle materials such as glasses or intrinsically brittle alloys at low temperatures, the calculation of $K(t)$ and the measurement of K_{ID}, which usually requires accurate determination of the crack velocity, can be difficult. For example, the basis of the continuum theory that the crack velocity depends only on the singular part of the crack-tip stress field can be questionable [8]. Currently, this issue is often analyzed using lattice models, in which the materials are considered as network of mass points connected by nonlinear springs [9], [10].

Based on these studies as well as the continuum mechanics analyses, it is generally accepted that for subsonic crack propagation the crack growth driving force can still be captured by the energy release rate, even though the crack tip stress field is distorted. This is equivalent to assuming that the energy dissipation in the near-tip field associated with the work of separation and the stress wave emission can be continuously balanced by the "release" of the strain energy from the background. Furthermore, since the characteristic time of thermal vibration of atoms is smaller than that of wave propagation by orders of magnitude, the surface free energy is independent of the crack velocity.

2 Governing equations

In order to avoid complicated simulations of the stress field at a dynamic crack tip, in this paper we will discuss the unstable crack propagation and the crack arrest through an energy analysis. For the sake of simplicity, we consider the contoured constant-K sample of a homogeneous, isotropic, and brittle material depicted in Fig. 1. The height of arms, $h(c)$, is given by [11]

$$\frac{4c^2}{h^3}+\frac{1+v}{h}=m, \tag{2}$$

where c is the distance to the loading axis, v is Poisson's ratio, and m is an arbitrarily chosen number. It will be shown shortly that the sample geometry has little influence on the fracture resistance. When $m \to 0$, h tends to infinity and the sample converges to an infinitely large plate.

In the constant-K sample, the effective compliance, C_e, is linear to the crack length [11],

$$C_e=\frac{\delta}{P}=\frac{6ma}{E}, \tag{3}$$

where δ is the crack opening displacement, P is the crack opening load per unit thickness, and E is the modulus of elasticity. Hence, with constant P the energy release rate is independent of the crack length. Noticing that $U=\delta P/2$ and $G=-\partial U/\partial a$, we have

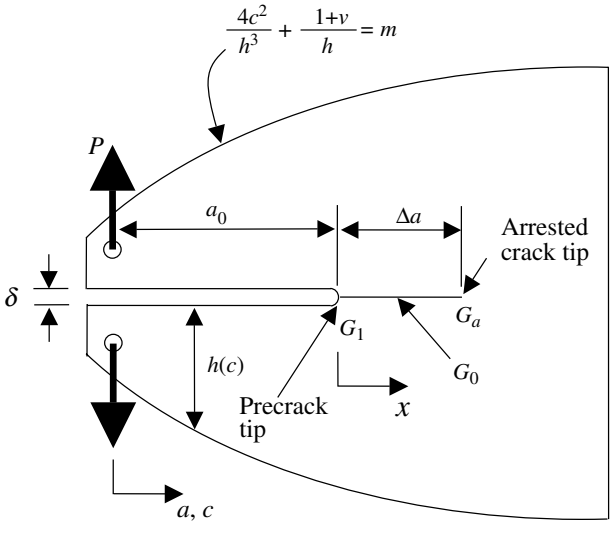

Fig. 1. A schematic diagram of the cleavage cracking in a constant-K sample

$$G = \frac{E}{12m}\frac{\delta^2}{a^2} \tag{4}$$

$$\text{and} \quad U = a\,G, \tag{5}$$

where U is the elastic energy per unit thickness.

The radius of the precrack tip can be arbitrarily chosen such that the critical energy release rate, G_1, at the onset of crack advance is considerably larger than the surface free energy, G_0. In the following discussion we will assume that G_0 and G_1 are given. As shown in Eq. (4), with the quasi-static increase in the crack opening displacement, the energy release rate rises rapidly. When G_1 is reached, the crack starts to propagate along the median plane. Associated with the increase in crack length, a certain amount of elastic energy is dissipated due to the work of separation of the fracture surfaces. As discussed above, with the assumption that the material is purely brittle, G_0 is independent of a and $\dot{a}$ [12]. Note that this assumption does not violate the fact that K_{ID} is a function of $\dot{a}$, since K_{ID} is related to the stress field and G_0 is determined by the atomic bonding, and in a dynamic fracture the ordinary relation of $K = \sqrt{E\,G/(1-\nu^2)}$ is no longer valid.

Because $G_1 > G_0$, the crack advance is unstable. During the dynamic crack propagation the variation in crack opening distance, δ , is negligible and, therefore, the static energy release rate keeps decreasing. Eventually, when G decreases to the critical value of crack arrest, G_{arrest}, the crack stops. When the crack growth distance is smaller than the initial crack length, $\dot{a}$ is lower than the sound speed, and the energy dissipation is balanced by the decrease in strain energy. Under this condition we have [13]

$$W - \frac{\mathrm{d}T}{\mathrm{d}A} - \frac{\mathrm{d}U}{\mathrm{d}A} = G_0 + D_{\mathrm{C}}, \tag{6}$$

where $W = \int\limits_S p_i \frac{\mathrm{d}u_i}{\mathrm{d}A}\,\mathrm{d}S + \int\limits_V b_i \frac{\mathrm{d}u_i}{\mathrm{d}A}\,\mathrm{d}V$ reflects the work done on the crack-tip field, with S being the integration surface at the crack tip, A the fracture area, p_{i} the surface traction, u_{i} the displacement, V the enclosed volume, and b_{i} the body force; T is the kinetic energy associated with the stress waves emitted from the crack tip; and D_{C} captures the continuum dissipation. Since δ is constant, $W = 0$. Thus, Eq. (6) becomes

$$\Delta U = D + T + G_0 x, \tag{7}$$

where ΔU is the difference between the elastic energy per unit thickness of current and initial configuration, x is the crack jump length, and $D = \int_0^x D_{\mathrm{C}}\,\mathrm{d}x$ is the accumulated dissipated energy per unit thickness.

Through Eq. (4), it can be obtained that

$$\tilde{G} = (a_1/a_0)^2 = (1+\Delta\tilde{a})^2 \tag{8}$$

with $\tilde{G} = G_1/G_{\mathrm{arrest}}$, $a_1 = a_0 + \Delta a$ the final crack length, and $\Delta\tilde{a} = \Delta a/a_0$. Similarly, for G_1 we have $\tilde{G}_1^* = (1+\tilde{x})^2$, where $\tilde{G}_1^* = G_1/G_x$, G_x is the static energy release rate when the crack growth length is x, and $\tilde{x} = x/a_0$. Note that $\tilde{G}$ is independent of $\tilde{x}$. Hence, according to Eq. (5), $\Delta U = a_0\,G_1 - (a_0 + x)\,G_{\mathrm{x}}$, which can be rewritten as

$$\Delta\tilde{U} = \frac{x\,\tilde{G}}{1+\tilde{x}}, \tag{9}$$

where $\Delta\tilde{U} = \Delta U/(a_0\,G_{\mathrm{a}})$. Substitution of Eq. (9) into (7) gives

$$\tilde{D} + \tilde{T} = \frac{\tilde{x}\,\tilde{G}}{1+\tilde{x}} - \beta\tilde{G}\tilde{x}, \tag{10}$$

where $\tilde{D} = D/(a_0\,G_{\mathrm{a}})$, $\tilde{T} = T/(a_0\,G_{\mathrm{a}})$, and $\beta = G_0/G_1 \le 1$.

As the crack advances, a considerable amount of kinetic energy is radiated from the crack tip into the background in the form of stress waves. The kinetic energy will be dissipated rapidly through damping, especially in materials of relatively high damping ratios such as epoxy resin and some BCC metals. Even if the characteristic time of wave decay is comparable with or even longer than that of crack growth, for large samples where the influence of wave reflection is negligible, the energy carried by the stress waves can still be considered as a part of the dissipated energy since it has little effect on the crack front behavior. If all the energy carried by the stress waves is eventually dissipated and the quasi-static crack-tip plastic deformation is negligible, $\tilde{D}$ can be related to $\tilde{T}$ through $\tilde{D} = \int_0^{\tilde{x}} \tilde{T}(\hat{x})\,\mathrm{d}\hat{x}$ or, equivalently, $\tilde{T} = \frac{\mathrm{d}\tilde{D}}{\mathrm{d}\tilde{x}}$.

Consequently, Eq. (10) can be rewritten as

$$\frac{\mathrm{d}\tilde{D}}{\mathrm{d}\tilde{x}} + \tilde{D} = \frac{\tilde{x}\,\tilde{G}}{1+\tilde{x}} - \beta\tilde{G}\tilde{x}, \tag{11}$$

with the boundary condition of $\tilde{D}\big|_{\tilde{x}=0} = 0$. Note that the factor of m vanishes in Eq. (11), indicating that $\tilde{D}$ is independent of the sample geometry.

3 Results and discussion

Equation (11) has the analytical solution

$$\tilde{D} = \tilde{G} - \beta\tilde{G}(\tilde{x}-1) - \tilde{G}e^{-(\tilde{x}+1)}\ln(\tilde{x}+1) - \tilde{G}(1+\beta)e^{-\tilde{x}} - \tilde{G}e^{-(\tilde{x}+1)}\hat{E}(\tilde{x}+1), \tag{12}$$

where $\hat{E}(\tilde{x}) = \sum_{n=1}^{\infty} \frac{(\tilde{x})^n - n(\tilde{x})^{n-1} - 1}{n \cdot n!}$. The derivative of Eq. (12) with respect to $\tilde{x}$ gives the kinetic energy

$$\tilde{T} = -\beta\tilde{G} + \tilde{G}e^{-(\tilde{x}+1)}\ln(\tilde{x}+1) - \tilde{G}\frac{e^{-(\tilde{x}+1)}}{\tilde{x}+1} + \tilde{G}(1+\beta)e^{-\tilde{x}}$$

$$+ \tilde{G}e^{-(\tilde{x}+1)}\hat{E}(\tilde{x}+1) - \tilde{G}e^{-(\tilde{x}+1)}\frac{\mathrm{d}\hat{E}(\tilde{x}+1)}{\mathrm{d}\tilde{x}}. \tag{13}$$

When $\tilde{x} = \Delta\tilde{a}$, the crack velocity is zero, i.e.,

$$\tilde{T}\big|_{\tilde{x}=\Delta\tilde{a}} = 0. \tag{14}$$

Combination of Eqs. (8), (13), and (14) gives $\Delta\tilde{a}$ and $\tilde{G}$ as functions of β. The numerical results are shown in Figs. 2 and 3. The relationship between the crack growth length Δa and β can be regressed as

$$\frac{\Delta a}{a_0} = f(\beta) \tag{15}$$

with $f(\beta) = -0.993 + 1.104(1/\beta) - 0.116(1/\beta)^2 + 0.005(1/\beta)^3$, based on which, through Eq. (8), the critical energy release rate can be obtained as

$$\frac{G_\mathrm{a}}{G_0} = g(\beta), \tag{16}$$

where $g(\beta) = 1/\left[\sqrt{\beta} + \sqrt{\beta}f(\beta)\right]^2$. Equation (16) indicates that G_a is not a material constant. It varies with G_1. If $G_1 = G_0$, the crack growth is stable and, thus, the dynamic effect is negligible, which is reflected by $G_\mathrm{a} = G_0$. As G_1/G_0 increases from 1 to 3, the dissipation of kinetic energy results in the rapid decrease of G_a to about $0.5G_0$. This value is close to, but somewhat lower

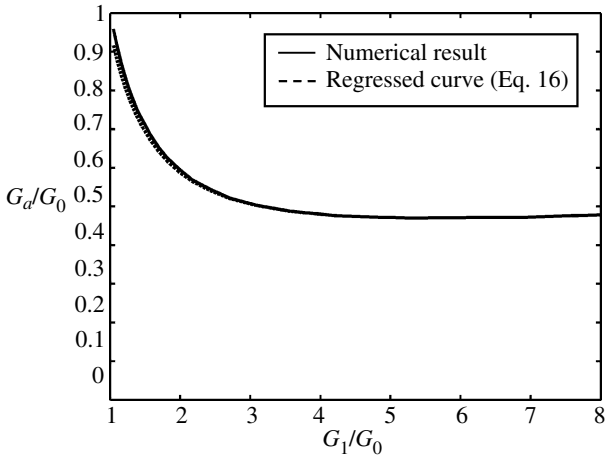

Fig. 2. The critical energy release rate of crack arrest, G_a, as a function of G_1/G_0, where $G_a/G_0 = 1/(\tilde{G}\beta)$ and $G_1/G_0 = 1/\beta$

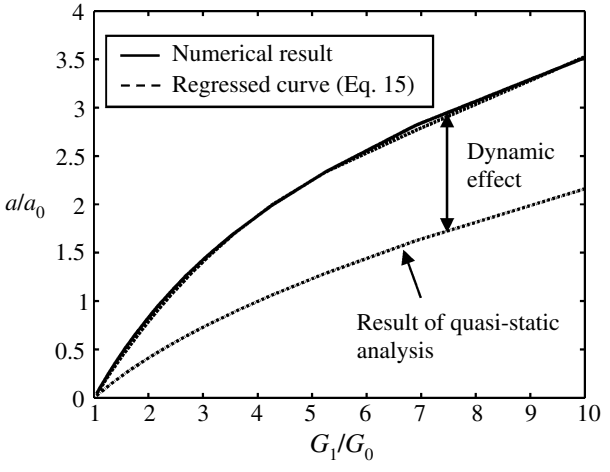

Fig. 3. The crack jump length as a function of G_1/G_0

than the experimental result of glasses [14], which can be attributed to that in the current case the propagating crack is assumed atomically sharp.

The dependence of G_a on G_1 demonstrates clearly the importance of the dynamic effect. At the tip of a dynamic crack, the energy must be balanced by the stress waves. The kinetic energy is the difference between the dynamic part of the crack growth driving force, $\Delta U - D$, and the nominal resistance, $G_0 x$. As long as the available energy for the stress wave emission is positive, the crack does not stop. This is quite different to the criterion of crack propagation dominated by the energy release rate itself. Therefore, the crack growth distance is longer than the result of the quasi-static analysis, $a_0\left(\sqrt{1/\beta} - 1\right)$, as shown in Fig. 3.

4 Conclusions

The criterion of crack arrest in brittle materials is discussed by an energy analysis for constant-K samples. As the crack length increases, the "released" elastic energy exceeds the work of separation and the dissipation. Thus the "extra" energy must be emitted into the background from the crack tip in the form of stress waves. When the available energy associated with the

inertial effect decreases to zero, the crack stops. As the ratio of G_1/G_0 increases to above 3, G_a decreases to about one half of the work of separation. This conclusion is valid for the cases where the crack growth rate is below the sound speed, so the requirement of energy conservation, Eq. (6), can be satisfied.

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