Modes of Loading

(1) – tension (a)
(2) – compression (b)
(3) – bending (c)
(4) – torsion (d)
and
combinations of them (e)
Standard Solution to Elastic Problems

Three common modes of loading:
(a) – tie with a circular cross-section loaded in tension
(b) – beam with rectangular cross-section loaded in bending
(c) – shaft of circular cross-section loaded in torsion

Relation between load, deflection and stiffness
\[ \delta = \frac{L_0 F}{AE} \]
\[ S = \frac{F}{\delta} = \frac{AE}{L_0} \]

Shape of cross-section does not matter because the stress is uniform across the section

Stiffness of material (a free variable – choice of material: a parameter that we need to find out)
Elastic Bending of Beams

Curvature of initial straight axis.......................... $\kappa = \frac{d^2u}{dx^2}$

Stress cause by bending moment in a beam with a given Young’s modulus............. $\frac{\sigma}{y} = \frac{M}{I} = E\kappa$

Second moment of inertia................................. $I = \int_{\text{section}} y^2 b(y) \, dy$

$u$: displacement parallel to y-axis  $M$: bending moment  $y$: distance from neutral axis

Cross-sectional Area and Second Moments for Four Shapes

<table>
<thead>
<tr>
<th>Section shape</th>
<th>Area $A$ $m^2$</th>
<th>Moment $I$ $m^4$</th>
<th>Moment $K$ $m^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$bh$</td>
<td>$\frac{bh^3}{12}$</td>
<td>$\frac{bh^3}{3} \left(1 - \frac{6b^2}{h^2}\right)$ $(b &gt; 0)$</td>
</tr>
<tr>
<td></td>
<td>$\pi r^2$</td>
<td>$\frac{\pi r^4}{4}$</td>
<td>$\frac{\pi r^4}{2}$</td>
</tr>
<tr>
<td></td>
<td>$\approx 2\pi rt$</td>
<td>$\approx \pi r^2 t$</td>
<td>$\approx \pi r^2 t$</td>
</tr>
<tr>
<td></td>
<td>$2t(h + b)$ $(h, b &gt;&gt; t)$</td>
<td>$\frac{1}{6}b^4(1 + \frac{5b}{h})$</td>
<td>$\frac{2bh^2p^2}{(b + h)}(1 - \frac{t}{h})^4$</td>
</tr>
</tbody>
</table>
Stiffness of a beam under transverse loading

\[ \delta = \frac{FL^3}{C_1EI} \]

\[ S = \frac{F}{\delta} = \frac{C_1EI}{L^3} \]

\[ C_1 \] is the only value that depends on the distribution of the load

Buckling of Columns and Plates

If sufficiently slender, an elastic column or plate, loaded in compression, fails by elastic buckling at a critical load

\[ F_{\text{crit}} = \frac{n^2 \pi^2 EI}{L^2} \]

Value of \( n \) depends on the end constraints of the beam
Torsion of Shafts

\( \tau: \) shear stress
\( r: \) radial distance from axis of symmetry
\( K: \) resistance to twisting (torsional equivalent of I)
\( G: \) shear modulus = \((3/8)E\) when \( \nu = 1/3 \)
\( T/\theta: \) torsional rigidity

\[
J = \int_{\text{section}} 2\pi r^3 \, dr
\]

\[
\frac{\tau}{r} = \frac{T}{K} = \frac{G\theta}{L}
\]

\[
\theta = \frac{TL}{GK} \quad S_T = \frac{T}{\theta} = \frac{GK}{L}
\]

Vibrating of Beams and Plates

An undamped system vibrating at its natural frequency can be expressed as the simple problem of a mass, \( m \), attached to a spring of stiffness, \( k \)

\[
f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}
\]

Lowest natural frequency of system

\[
f = \frac{C_2}{2\pi} \sqrt{\frac{EI}{m_0L^4}}
\]

Natural frequency based on spring stiffness, \( k \), of a beam in bending

\[
m_0 \text{ is the mass per unit length or (area x density); therefore, frequencies scale with } (E/\rho)^{1/2}
\]
Material Indices for Elastic Design

Minimizing Weight

A light, stiff tie-rod:

Design Requirements

Constraints
- Length, $L_0$, is specified
- Cross-sectional area must be $A_0$
- Must carry a tensile force $F_0$ without extending elastically by more than $\delta_0$
  \[ S^* = \frac{F_0}{\delta_0} \]
- Must have some toughness

Objective
- Should be as lightweight as possible

Free Variable: Choice of Materials

In this course, choice of material ($\rho$, $E$, $\sigma_y$, etc.) is always a free variable
Compare to: No free variable – might not have any solution!

Constraints: \( S^*, L_0, A \), must be 6061 Al alloy…

Objective: Mass \((m)\)

Free variables: N/A

Or: Only one free variable – group all the materials parameters in a single item, based on which materials candidates can be ranked

Constraints: \( S^*, L_0, A \rightarrow \) Eliminate irrelevant choices

Objective: Mass \((m)\) \rightarrow Rank relevant candidates

Free variables: Choice of material (toughness, mass density…)

Complicated case: There are 2 or more free variables affecting both constraints and objective \(\rightarrow\) Must derive the performance index (discussed later)

Objective function: equation that describes the quantity to be maximized or minimized

\[ m = A L_0 \rho \]

The goal is to minimize the value of the objective function within the given constraints

Constraint: Section area \(A\) must be sufficient to provide a stiffness of \(S^*\)

\[ S^* = \frac{AE}{L_0} \]
One Free Variable

If there were only one free variables (an “easy” problem):
Constraints: \( S^* \) (stiffness of rod), \( L_0, A_0 \) \( \rightarrow \) Eliminate irrelevant choices
Objective: Mass \((m)\) \( \rightarrow \) Rank relevant candidates
Free variables: Choice of material

1) Use constraints to screen:
\[ E^* = \frac{S^* L_0}{A_0} \] (e.g. 10 GPa)

2) Use objective to rank

Optimum choice: natural material (wood)

Minimizing Weight

A light, stiff tie-rod:

**Design Requirements**

**Constraints**
- Length, \( L_0 \), is specified
- Cross-sectional area must be \( A_0 \)
- Must carry a tensile force \( F_0 \) without extending elastically by more than \( \delta_0 \)
  \( \Rightarrow \) Stiffness must be at least \( S^* = \frac{F_0}{\delta_0} \)
- Must have some toughness

**Objective**
- Should be as lightweight as possible

**Free Variable:** Choice of Materials
Cross-sectional area, \( A \)
Eliminate the free variable \( A \) to obtain:

\[
m = S^* L_o^2 \left( \frac{\rho}{E} \right)
\]

The smaller the mass density \( \rho \), the larger the performance index, \( M_t \) (the smaller the total mass, \( m \)).

Two free variables; one of them should be “eliminated”

Objective function

There must be at least one free variable (degree of freedom); otherwise the design is impossible.

Eliminate the free variable \( A \) to obtain:

\[
m = S^* L_o^2 \left( \frac{\rho}{E} \right)
\]

The larger the Young’s modulus \( E \), the smaller the cross-sectional area \( A \) needs to be to reach the required \( S^* \), resulting in a smaller total mass, \( m \).

Performance Index for material selection

High values of \( M_t \) are the best choice; the function \( E/\rho \) is called the specific stiffness.
\[ M_t = \frac{E}{\rho} \]

Constant line for \( M_t \) or \( \log(M_t) \) is:

\[
\log(E) - \log(\rho) = \log(M_{t0})
\]

or

\[
\log(E) = \log(\rho) + \log(M_{t0})
\]

or

\[ y = x + c \text{ in the left figure} \]

In the family of parallel lines of \( y = x+c \), the higher the line, the larger the value of \( M_{t0} \).

But it cannot exceed the area of bubbles.

If: When the force is \( F_{\max} \), the elongation must be close to \( \Delta L_0 \);

\([A] \text{ must be in the range of } A_0 \pm \delta A;\)

\(L\) is a free variable

\[
m = A^2 \left( \frac{E}{\rho} \right) / S^* \]

The material performance index becomes

\[ M_t = (E\rho) \]

\( M_t \) collectively sets the criterion of material selection

A smaller \( E \) leads to a shorter rod (and thus, a lower mass), to achieve \( S^* \)

A smaller \( \rho \) leads to a lower mass, \( m \)
Two Free Variables

Constant line for $M_t$ or $\log(M_t)$ is:

$$\log(E) + \log(\rho) = \log(M_{t0})$$

or

$$\log(E) = -\log(\rho) + \log(M_{t0})$$

or

$$y = -x + c$$ in the left figure

In the family of parallel lines of $y = -x + c$, the higher the line, the larger the value of $M_{t0}$.

But it cannot exceed the area of bubbles.

Minimizing Weight: A light, stiff panel

Design Requirements

- Length $L$ and width $b$ are specified
- Thickness $h$ is free
- Loaded by bending by a central load $F$
- Stiffness constraint requires it must not deflect more than $\delta$ under the load
- Objective it to make the panel as light as possible
### Table 5.2

**Objective function:**

\[ m = AL\rho = bhL\rho \]

**Stiffness Constraint:**

\[ S^* = \frac{C_1 EI}{L^3} \]

Second moment of inertia for rectangular section:

\[ I = \frac{bh^3}{12} \]

---

**Material Index: Light, Stiff Panel**

Stiffness \( S^* \), length \( L \), and width \( b \) are specified; thickness \( h \) is free.

Eliminating \( h \) from the objective function gives:

\[ m = \left( \frac{12S^*}{C_1 b} \right)^{1/3} \left( bL^2 \right) \left( \frac{\rho}{E^{1/3}} \right) \]

Unspecified variables leaves \( \rho/E^{1/3} \) which would be minimized to minimize mass; as before the function will be inverted so a maximum value will be sought.

\[ M_p = \frac{E^{1/3}}{\rho} \]
Ranking: indices on charts

Selection lines are used based on the material indices. All materials that lie on the selection line perform equally well; those that lie above the line perform better.

For the selection of a light, stiff panel, the material index is $E^{1/3}/\rho$.

A material with $M = 3$ gives a panel that has one-tenth the weight of one with $M = 0.3$. 

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Minimizing Weight
A light, stiff beam

Objective function: \[ m = AL\rho = b^2 L\rho \]

Stiffness constraint and second moment of inertia:
\[ S^* = \frac{C_1 EI}{L^3} \quad I = \frac{b^4}{12} = \frac{A^2}{12} \]

Eliminating the free variable A from the objective function and identifying the unspecified variables:
\[ m = \left( \frac{12S^* L^3}{C_1} \right)^{1/2} (L) \left( \frac{\rho}{E^{1/2}} \right) \quad M_b = \frac{E^{1/2}}{\rho} \]
Ranking: indices on charts

Selection lines are used based on the material indices

All materials that lie on the selection line perform equally well; those that lie above the line perform better.

Shape Factor

By reshaping the cross-section of a beam, it is possible to increase \( I \) – thus increasing stiffness – without increasing the total area.

The ratio of \( I \) for the shaped section to that for a solid square section with the same area is defined as the shape factor \( \Phi \).
Shaping is also used to make structures lighter; it is a way to achieve the same stiffness with less material.

<table>
<thead>
<tr>
<th>Material</th>
<th>Typical maximum shape factor (stiffness relative to that of a solid square beam)</th>
<th>Typical mass ratio by shaping (relative to that of a solid square beam)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steels</td>
<td>64</td>
<td>1/8</td>
</tr>
<tr>
<td>Al alloys</td>
<td>49</td>
<td>1/7</td>
</tr>
<tr>
<td>Composites (GFRP, CFRP)</td>
<td>36</td>
<td>1/6</td>
</tr>
<tr>
<td>Wood</td>
<td>9</td>
<td>1/3</td>
</tr>
</tbody>
</table>

Shaping could increase the value of $I$ by 64X if steel was used in the construction of a beam with a square cross-section; if wood was used, the maximum increase would be 9X.

$S^* = \frac{C_1EI}{L^3}$

Case Studies: Lever for corkscrew

Lever is loaded in bending and acts as a light, stiff beam.

$M = \frac{E^{1/2}}{\rho}$

| Function Constraints                  |  |
|--------------------------------------|  |
| Objective                            |  |
| Free variables                       |  |
| Light weight lever, meaning light, stiff beam |  |
| Stiffness $S^*$ specified            |  |
| Length $L$                           |  |
| Section shape rectangular            |  |
| Minimise mass                        |  |
| Choice of material                   |  |
| Area $A$ of cross-section            |  |
Based on Figure 5.14, the materials most suited for a stiff, light corkscrew lever are ceramics, composites, and woods.

Other considerations such as cost will help narrow the selection further.

Minimizing Materials Cost

Material indices change as the objective changes; the objective function for minimizing cost of the tie rod, panel, or beam would be:

\[ C = m C_m = A L C_m \rho \]

With \( A \) (cross-sectional area) and \( L \) (length) being specified, the goal of a material selection would be to minimize \( C_m \rho \) or maximize \( 1 / C_m \rho \). It is equivalent to using a weight factor of \( C_m \) for the volume density (\( \rho \)) in the analysis of light, stiff beam.
Plotting Limits and Indices on Charts

Screening: attribute limits on charts

Constraints can be plotted as horizontal or vertical lines on material property charts

Constraints in Figure 5.9:
E > 10 GPa
Relative Cost < 3

All materials in the search region meet both constraints

Case Studies: Structural materials

A stiff beam with minimum cost is required

The analysis process is similar with that of a lightweight, stiff beam, except that the density is weighted by the materials cost per mass ($C_m$)

$$M = \frac{E^{1/2}}{\rho C_m}$$

~ Cost per unit volume, $C_V$

<table>
<thead>
<tr>
<th>Function Constraints</th>
<th>Objective Free variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Floor beam</td>
<td>Floor beam</td>
</tr>
<tr>
<td>Stiffness $S^*$ specified</td>
<td>Stiffness $S^*$ specified</td>
</tr>
<tr>
<td>Length $L$ specified</td>
<td>Length $L$ specified</td>
</tr>
<tr>
<td>Section shape square</td>
<td>Section shape square</td>
</tr>
<tr>
<td>Minimise material cost</td>
<td>Minimise material cost</td>
</tr>
<tr>
<td>Choice of material</td>
<td>Choice of material</td>
</tr>
<tr>
<td>Area $A$ of cross-section</td>
<td>Area $A$ of cross-section</td>
</tr>
</tbody>
</table>
Selection of stiff floor beam

Concrete, stone, and brick have poor tensile strength.

Wood, steel, and reinforced concrete have strength in tension and compression.

Steel is also has a higher shape factor allowing for greater freedom of the form of the building.

Selection of stiff bar stool shaft

What is the best material for the shaft in a bar stool?

Constraint: Cannot deform too much under external loading:

A. Tension?
B. Compression?
C. Bending?
D. Torsion?

Objective: minimize or maximize cost?
Bendy Design: Part stiff, Part-Flexible Structures

Components can be designed to be strong but not stiff

Elastic bending locates and guides the plunger as well as provides the restoring force provided by the spring

Rotational hinges is replaced by elastic connecting strip

Various section shapes, each allowing one or more degrees of freedom, while retaining stiffness and strength in the other directions